

An Efficient Convex-Hull Relaxation Based Algorithm for Multi-User Discrete Passive Beamforming

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Abstract—Intelligent reflecting surface (IRS) is an emerging technology to enhance spatial multiplexing in wireless networks. This letter considers the discrete passive beamforming design for IRS in order to maximize the minimum signal-to-interference-plus-noise ratio (SINR) among multiple users in an IRS-assisted downlink network. The main design difficulty lies in the discrete phase-shift constraint. Differing from most existing works, this letter advocates a convex-hull relaxation of the discrete constraints which leads to a continuous reformulated problem equivalent to the original discrete problem. This letter further proposes an efficient alternating projection/proximal gradient descent and ascent algorithm for solving the reformulated problem. Simulation results show that the proposed algorithm outperforms the state-of-the-art methods significantly.

Index Terms—Convex-hull relaxation, discrete passive beamforming, intelligent reflecting surface (IRS).

I. INTRODUCTION

INTELLIGENT reflecting surface (IRS) is an emerging wireless technology that aims to boost signal reception by coordinating the phase shifts of reflected paths [1], namely passive beamforming. In practice, the phase shift choice is typically limited to a prescribed discrete set. While the discrete passive beamforming for a single user can be efficiently solved [2], the multiple-user case (e.g., for industrial automation) is much more challenging and remains an open problem.

The main difficulty in the discrete passive beamforming design for the multi-user case lies in the discrete constraint. Most previous works [3]–[5] just ignore the discrete constraint at the optimization stage and then round the continuous solution to the discrete set. It is generally hard to justify this heuristic relaxation because the relaxed problem without the discrete constraint is not equivalent to the original problem. In sharp contrast to [3]–[5], this letter advocates a convex-hull relaxation that guarantees the equivalence to the original discrete problem. Further, this letter proposes an efficient alternating projection/proximal gradient descent and ascent algorithm for solving the relaxed problem. As shown in [6], the branch-and-bound method and the heuristic methods including the genetic algorithm have been considered.

Many previous efforts in the realm of passive beamforming focus on the single-user case. To maximize the spectral efficiency for a multiple-input multiple-output (MIMO) link, the work [7] coordinates the phase shifts of IRS by the Riemannian conjugate gradient method. Considering the double-IRS system, the work [8] proposes a geometric approach to the signal-to-noise ratio (SNR) maximization problem. The more recent work [9] extends the SNR problem to the case of multiple IRSs. While all the above works require the

full channel state information (CSI), a line of studies [10]–[12] advocate blind beamforming without using any channel knowledge. Specifically, [10] addresses a special ON-OFF case of passive beamforming, [11] solves the general K -ary passive beamforming problem approximately while [2] solves it globally, and [12] deals with multiple IRSs, all for maximizing the received signal power at a single target receiver.

For the multi-user case, some existing works consider the common-message multicast network [13]–[16]. To avoid channel estimation, the work [13] proposes an ad-hoc passive beamforming scheme that randomly configures the IRS during the channel coherence interval. Assuming that the CSI is available and the phase shift of each reflective element (RE) can be chosen arbitrarily, the work [15] proposes an alternating direction method of multipliers based method for the max-min SNR problem. In the presence of the discrete constraint, the work [16] suggests a gradient descent-ascent (GDA) approach. In contrast, [17]–[21] account for the downlink transmission with interference. More specifically, the work [17] aims to optimize two IRSs jointly in order to maximize the minimum signal-to-interference-plus-noise ratio (SINR) across multiple users, and [18] considers a weighted max-min SINR problem. Moreover, [19] considers maximizing the sum rate, [20] considers minimizing the total latency, and [21] considers maximizing the system energy efficiency. As for the used algorithms, all of [17]–[20] rely on the semidefinite relaxation method while [21] uses the Riemannian gradient method.

II. SYSTEM MODEL

Consider an IRS-assisted multi-user downlink network in which the base station (BS) with M antennas serves a total of U single-antenna users. The IRS comprises N REs. We use $u = 1, 2, \dots, U$ to index the users, and use $n = 1, 2, \dots, N$ to index the REs. Denote by $\mathbf{d}_u \in \mathbb{C}^M$ the straight channel from the BS to user u . Denote by $\mathbf{F} \in \mathbb{C}^{N \times M}$ the channel from the BS to the IRS, whose (n, m) th entry is the channel from the m th antenna of the BS to the n th RE. Denote by $\mathbf{g}_u \in \mathbb{C}^{1 \times N}$ the channel from the IRS to user u , whose n th entry is the channel from the n th RE to user u . Let $\mathbf{w}_u \in \mathbb{C}^M$ be the beamforming vector at the BS for user u ; write $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_U]$ with the power constraint $\|\mathbf{W}\|_F^2 \leq P$.

We denote by $\theta_n \in [0, 2\pi)$ the phase shift induced by RE n in its corresponding reflected path. From a practical standpoint, we further assume that each θ_n can only take on values from the discrete set $\{0, \frac{2\pi}{K}, 2 \times \frac{2\pi}{K}, \dots, (K-1) \times \frac{2\pi}{K}\}$ given some integer $K \geq 2$. Define the variable

$$\mathbf{x} = [1, x_1, x_2, \dots, x_N]^H \text{ where each } x_n = e^{j\theta_n}. \quad (1)$$

Because of the discrete constraint on θ_n , each x_n is limited to

$$\mathcal{X} = \left\{ e^{j\frac{2k\pi}{K}} \mid k = 0, 1, \dots, K-1 \right\}. \quad (2)$$

With the $(N+1) \times M$ matrix

$$\Phi_u = \begin{bmatrix} \mathbf{d}_u^\top \\ \text{diag}(\mathbf{g}_u) \mathbf{F} \end{bmatrix}, \quad (3)$$

the received signal at user u can be computed as $\mathbf{x}^H \Phi_u \mathbf{w}_u s_u + \sum_{u'=1, u' \neq u}^U \mathbf{x}^H \Phi_u \mathbf{w}_{u'} s_{u'} + z_u$, where $s_u \sim \mathcal{CN}(0, 1)$ is the independent symbol intended for user u , and $z_u \sim \mathcal{CN}(0, \sigma_u^2)$ is the background noise. The resulting SINR of user u is

$$\text{SINR}_u = \frac{\mathbf{x}^H \mathbf{C}_{uu} \mathbf{x}}{\sum_{u' \neq u} \mathbf{x}^H \mathbf{C}_{uu'} \mathbf{x} + \sigma_u^2}, \quad (4)$$

where

$$\mathbf{C}_{uu'} = \Phi_u \mathbf{w}_{u'} \mathbf{w}_{u'}^H \Phi_u^H. \quad (5)$$

To achieve the max-min fairness for U downlink users, we consider the joint active and passive beamforming problem as

$$\underset{\mathbf{W}, \mathbf{x}}{\text{maximize}} \quad \min_u \{ \text{SINR}_u \} \quad (6a)$$

$$\text{subject to} \quad x_n \in \mathcal{X}, \quad n = 1, 2, \dots, N, \quad (6b)$$

$$\|\mathbf{W}\|_F^2 \leq P. \quad (6c)$$

We propose optimizing two variables \mathbf{W} and \mathbf{x} alternately (e.g., as in [18] and [22]). In particular, when \mathbf{x} is fixed, the problem of \mathbf{W} can be optimally solved as in [23].

The rest of the letter focuses on optimizing \mathbf{x} in problem (6) with \mathbf{W} being fixed. The main difficulty in optimizing \mathbf{x} lies in the discrete constraint (6b). To overcome this difficulty, many previous works [3]–[5] propose to simply ignore the discrete constraint by allowing each phase shift θ_n to take an arbitrary value in $[0, 2\pi)$, i.e., \mathcal{X} is relaxed as

$$\hat{\mathcal{X}} = \{ e^{j\theta} \mid \theta \in [0, 2\pi) \} = \{ x \in \mathbb{C} \mid |x| = 1 \}. \quad (7)$$

Hence, each x_n now lies on the unit circle in the complex plane. After the relaxed problem is solved, the obtained solution is rounded to the discrete set \mathcal{X} in (2). However, such relaxation cannot guarantee the equivalence between the relaxed problem and the original one, thereby resulting in a potentially large performance loss.

III. CONVEX-HULL RELAXATION

Our work also seeks to eliminate the discrete constraint from (6), but by relaxing the discrete set \mathcal{X} to its convex hull

$$\text{conv}(\mathcal{X}) = \left\{ \sum_{k=1}^K \lambda_k x_k \mid x_k \in \mathcal{X}, \sum_{k=1}^K \lambda_k = 1, \lambda_k \geq 0, \forall k \right\}. \quad (8)$$

Fig. 1 shows the conventional relaxation $\hat{\mathcal{X}}$ in (7) and the convex-hull relaxation $\text{conv}(\mathcal{X})$ in (8) of the discrete set \mathcal{X} when $K = 4$. The main feature of the proposed convex-hull relaxation $\text{conv}(\mathcal{X})$ is that it allows for an equivalent continuous reformulation of (6), which significantly facilitates the algorithmic development. As shown in Fig. 1, all the feasible solutions in \mathcal{X} turn out to be the vertices of the convex

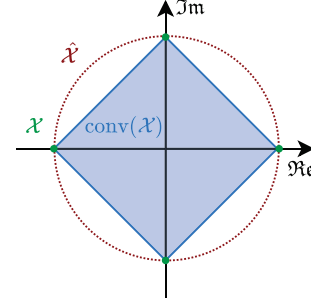


Fig. 1. Conventional relaxation $\hat{\mathcal{X}}$ vs. convex-hull $\text{conv}(\mathcal{X})$ when $K = 4$.

hull $\text{conv}(\mathcal{X})$, which are the farthest points from the origin in the complex unit ball. As such, it will encourage choosing a discrete solution from \mathcal{X} if we require the solution to be sufficiently far away from the origin.

The above idea can be realized by reformulating the discrete problem (6) (with respect to \mathbf{x}) as a continuous problem:

$$\underset{\mathbf{x}}{\text{maximize}} \quad \min_u \left\{ \frac{\mathbf{x}^H \mathbf{C}_{uu} \mathbf{x}}{\sum_{u' \neq u} \mathbf{x}^H \mathbf{C}_{uu'} \mathbf{x} + \sigma_u^2} \right\} + \lambda \|\mathbf{x}\|_1 \quad (9a)$$

$$\text{subject to} \quad x_n \in \text{conv}(\mathcal{X}), \quad n = 1, 2, \dots, N, \quad (9b)$$

where $\lambda > 0$ is a positive parameter. Intuitively, by increasing λ , the solution of (9) would go toward a vertex of $\text{conv}(\mathcal{X})$. Clearly, as $\lambda \rightarrow \infty$, the solution of (9) must be a vertex of $\text{conv}(\mathcal{X})$ and hence it must lie in \mathcal{X} . One main result of this letter is to show that it suffices to use a finite parameter λ to achieve the above goal, i.e., the relaxed problem (9) with a proper λ is equivalent to the original discrete problem (6).

Lemma 1: All locally and globally optimal solutions of problem (9) lie on the boundary of $\text{conv}(\mathcal{X})$, which is denoted as $\partial \text{conv}(\mathcal{X})$, so long as the parameter

$$\lambda > \max_u L_u, \quad (10)$$

where $L_u = \frac{2\|\mathbf{C}_{uu}\|_F}{\sigma_u^2} \left(\sum_{u' \neq u} \frac{(N+1)\|\mathbf{C}_{uu'}\|_F}{\sigma_{u'}^2} + 2 \right)$.

Proof: Suppose that there exists a locally or globally optimal solution \mathbf{x} of (9) with its entry $x_{n'} \notin \partial \text{conv}(\mathcal{X})$. Then, for this $x_{n'}$, we can find a real scalar $r > 1$ such that $rx_{n'} \in \partial \text{conv}(\mathcal{X})$, as illustrated in Fig. 2(a). For any real scalar $1 < \delta < r$, define

$$\mathbf{z}^\delta = [1, z_1^\delta, \dots, z_N^\delta]^\top \text{ where } z_n^\delta = \begin{cases} \delta x_n, & \text{if } n = n'; \\ x_n, & \text{otherwise.} \end{cases} \quad (11)$$

Moreover, consider the function

$$f_u(\mathbf{x}) = \frac{\mathbf{x}^H \mathbf{C}_{uu} \mathbf{x}}{\sum_{u' \neq u} \mathbf{x}^H \mathbf{C}_{uu'} \mathbf{x} + \sigma_u^2}. \quad (12)$$

It can be shown that

$$f_u(\mathbf{x}_1) - f_u(\mathbf{x}_2) \geq -L_u \|\mathbf{x}_1 - \mathbf{x}_2\|_2 \quad (13)$$

for any two $\mathbf{x}_1, \mathbf{x}_2 \in \text{conv}(\mathcal{X})$, so L_u is the Lipschitz constant of $f_u(\mathbf{x})$. We then have

$$f_u(\mathbf{z}^\delta) - f_u(\mathbf{x}) + \lambda \|\mathbf{z}^\delta\|_1 - \lambda \|\mathbf{x}\|_1 \geq (\lambda - L_u) (\delta - 1) |x_{n'}|. \quad (14)$$

The relaxed optimization objective in (9a) can be written as $f_0(\mathbf{x}) = \min_u \{f_u(\mathbf{x})\} + \lambda \|\mathbf{x}\|_1$. It immediately follows from (10) and (14) that $f_0(\mathbf{z}^\delta) - f_0(\mathbf{x}) > 0$ for all $\delta \in (1, r)$. This shows that \mathbf{x} is not a locally optimum solution, so we arrive at a contradiction. The proof is then completed. ■

Theorem 1: The relaxed problem (9) is equivalent to the original discrete problem (6) if the parameter

$$\lambda > \frac{\sin(\pi/K)}{1 - \cos(\pi/K)} \cdot \max_u L_u. \quad (15)$$

Proof: We start by showing that a solution \mathbf{x} of problem (9) must be a solution of problem (6). The most nontrivial step is to show that each entry of this solution \mathbf{x} lies in the discrete set \mathcal{X} . Since any λ in (15) must satisfy the condition in (10), \mathbf{x} must lie on the convex-hull boundary $\partial \text{conv}(\mathcal{X})$ according to Lemma 1. Assume that \mathbf{x} has an entry $x_{n'} \notin \mathcal{X}$ and $v \in \mathcal{X}$ is the nearest discrete point, as shown in Fig. 2(b). We let

$$\mathbf{z} = [1, z_1, \dots, z_N]^\top \text{ where } z_n = \begin{cases} v, & \text{if } n = n'; \\ x_n, & \text{otherwise.} \end{cases} \quad (16)$$

Based on (13), we further derive

$$f_u(\mathbf{z}) - f_u(\mathbf{x}) + \lambda \|\mathbf{z}\|_1 - \lambda \|\mathbf{x}\|_1 \geq -L_u |v - x_{n'}| + \lambda(1 - |x_{n'}|). \quad (17)$$

Let $t = |v - x_{n'}|$; observe that $0 < t \leq \sin(\pi/K)$. With K and t , the value of $|x_{n'}|$ is given by

$$|x_{n'}| = \sqrt{\cos^2\left(\frac{\pi}{K}\right) + \left(\sin\left(\frac{\pi}{K}\right) - t\right)^2}. \quad (18)$$

Substituting (18) into (17), we obtain

$$f_u(\mathbf{z}) - f_u(\mathbf{x}) + \lambda \|\mathbf{z}\|_1 - \lambda \|\mathbf{x}\|_1 \geq -L_u \cdot t + \lambda - \lambda \sqrt{\cos^2\left(\frac{\pi}{K}\right) + \left(\sin\left(\frac{\pi}{K}\right) - t\right)^2}. \quad (19)$$

Observe that when \mathbf{z} and \mathbf{x} are both fixed, there exists some $t_0 \in (0, \sin(\pi/K)]$ such that the lower bound in the right-hand side of (19) increases with t for $t \in (0, t_0)$ and decreases with t for $t \in [t_0, \sin(\pi/K)]$. Furthermore, due to (15), the infimum of the lower bound must be attained at $t = 0$, so it holds that $f_u(\mathbf{z}) - f_u(\mathbf{x}) + \lambda \|\mathbf{z}\|_1 - \lambda \|\mathbf{x}\|_1 > 0$ for any $t \in (0, \sin(\pi/K)]$. We still write the objective function of (9) as $f_0(\mathbf{x}) = \min_u \{f_u(\mathbf{x})\} + \lambda \|\mathbf{x}\|_1$. It follows that $f_0(\mathbf{z}) > f_0(\mathbf{x})$. The above result contradicts the assumption that \mathbf{x} is globally optimal for problem (9). Therefore, as long as \mathbf{x} is a globally optimal solution of (9), we must have $x_n \in \mathcal{X}$ for all n and $\|\mathbf{x}\|_1 = N + 1$. Now it is simple to show that the solution \mathbf{x} of (9) must be a solution of (6). For the contradiction purpose, assume that \mathbf{x} is not optimal for (6) while $\tilde{\mathbf{x}}$ is a globally optimum solution of (6). Thus, we must have $\min_u \{f_u(\tilde{\mathbf{x}})\} > \min_u \{f_u(\mathbf{x})\}$. Moreover, because $\|\tilde{\mathbf{x}}\|_1 = \|\mathbf{x}\|_1 = N + 1$, we further have $f_0(\tilde{\mathbf{x}}) > f_0(\mathbf{x})$; which contradicts the assumption that \mathbf{x} is globally optimal for (9). Thus, \mathbf{x} must be a globally optimal solution of (6).

We then verify the converse, i.e., a solution \mathbf{x} of (6) must be a solution of (9). Suppose that $\tilde{\mathbf{x}}$ is a solution of (9). From the

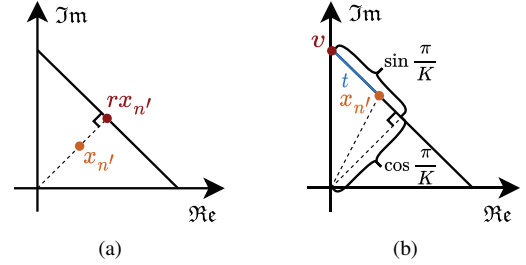


Fig. 2. Two key steps in proving Lemma 1 and Theorem 1. Panel (a) shows the definition of $rx_{n'}$. Panel (b) illustrates the computation of $|x_{n'}|$.

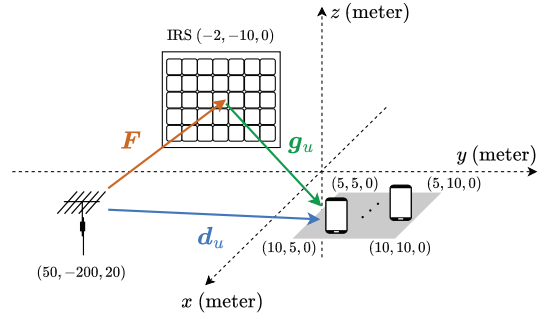


Fig. 3. The IRS-assisted downlink network in our simulations.

previous part, we know that $\tilde{\mathbf{x}}$ must be a solution (6). Since \mathbf{x} and $\tilde{\mathbf{x}}$ are both feasible for (6), it follows that $\|\mathbf{x}\|_1 = \|\tilde{\mathbf{x}}\|_1 = N + 1$; since \mathbf{x} and $\tilde{\mathbf{x}}$ are both globally optimal for (6), it holds that $\min_u \{f_u(\tilde{\mathbf{x}})\} = \min_u \{f_u(\mathbf{x})\}$; thus, \mathbf{x} is equally good as $\tilde{\mathbf{x}}$ for (9), so \mathbf{x} must be a globally optimal solution of (9). Combining the above results verifies the equivalence. ■

Here are two remarks on Lemma 1 and Theorem 1. First, the use of $\|\mathbf{x}\|_1$ in (9) is critical. If we use $\|\mathbf{x}\|_2^2$ instead, then it is difficult to show that all the locally optimal points must lie on the convex hull boundary. Further, if we switch to some other penalty terms, then we are faced with the problem of whether a finite lower bound on λ still exists. Second, using the convex hull to approximate the discrete set can be found in the literature [24], [25], but its application to the passive beamforming has not yet been explored. Moreover, because the objective function in (6) is more complicated than that in [24], it requires new efforts to establish the equivalence to the original problem, as shown in Lemma 1 and Theorem 1.

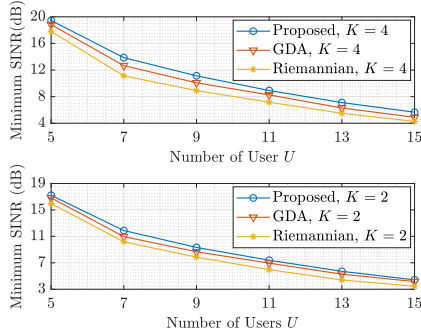
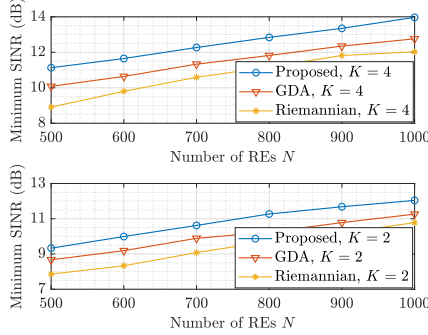
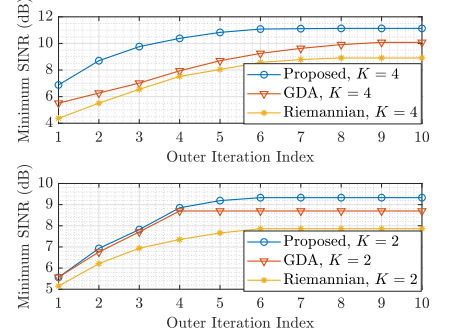
IV. PROPOSED ALGORITHM

In this section, we propose an alternating projection/proximal gradient descent and ascent algorithm for solving problem (9). The proposed algorithm is based on the following reformulation of problem (9):

$$\max_{\mathbf{x}} \min_{\mathbf{y} \in \Delta} \sum_{u=1}^U \frac{y_u \mathbf{x}^H \mathbf{C}_{uu} \mathbf{x}}{\sum_{u' \neq u} \mathbf{x}^H \mathbf{C}_{u'u} \mathbf{x} + \sigma_u^2} + \lambda \|\mathbf{x}\|_1 \quad (20a)$$

$$\text{subject to } x_n \in \text{conv}(\mathcal{X}), n = 1, 2, \dots, N, \quad (20b)$$

where $\mathbf{y} = [y_1, y_2, \dots, y_U]^\top$ is an auxiliary variable and $\Delta = \{\mathbf{y} \mid \mathbf{1}^\top \mathbf{y} = 1, y_u \geq 0\}$ is the simplex. The equivalence between (9) and (20) follows from the following observation:

Fig. 4. Minimum SINR vs. U when $N = 500$.Fig. 5. Minimum SINR vs. N when $U = 9$.Fig. 6. Convergence as $N = 500$ and $U = 9$.

Algorithm 1 Proposed Algorithm

- 1: **input:** $\{\Phi_u\}$ and \mathcal{X} .
 - 2: **for** $i = 1, 2, \dots, T_1$ **do**
 - 3: Decide \mathbf{W} by the method in [23].
 - 4: initialize $\lambda = 0.0001$.
 - 5: **for** $j = 1, 2, \dots, T_2$ **do**
 - 6: **if** $\text{mod}(j, 100)$ is 0 **then**
 - 7: Update $\lambda = 10\lambda$.
 - 8: **end if**
 - 9: Update \mathbf{y} by (21).
 - 10: Update \mathbf{x} by (22).
 - 11: **end for**
 - 12: **end for**
 - 13: **Output:** \mathbf{W} and \mathbf{x} .
-

if \mathbf{x} has been optimally determined, we would assign $y_u = 1$ to the minimum term while setting $y_{u'} = 0$ for the rest.

We propose optimizing \mathbf{x} and \mathbf{y} alternately in (20). Denote by $g(\mathbf{x}, \mathbf{y})$ the first smooth term and denote by $h(\mathbf{x})$ the second nonsmooth term of the optimization objective in (20a), respectively. For fixed \mathbf{x} , the auxiliary variable \mathbf{y} can be updated by performing a projection gradient descent step:

$$\mathbf{y}^{t+1} = \text{Proj}_{\Delta}(\mathbf{y}^t - \alpha^t \nabla_{\mathbf{y}} g(\mathbf{x}^t, \mathbf{y}^t)), \quad (21)$$

where t is the iteration index, $\alpha^t > 0$ is the stepsize, and $\text{Proj}_{\Delta}(\cdot)$ is the projection onto Δ . The simplex projection can be efficiently done with a complexity of $O(U \log U)$ [26]. For fixed \mathbf{y} , update \mathbf{x} by the proximal gradient ascent:

$$\mathbf{x}^{t+1} = \arg \min_{\mathbf{x} \in \mathcal{A}} \frac{1}{2\beta^t} \|\mathbf{x} - \mathbf{x}^t - \beta^t \nabla_{\mathbf{x}} g(\mathbf{x}^t, \mathbf{y}^{t+1})\|_2^2 - h(\mathbf{x}), \quad (22)$$

where $\beta^t > 0$ and $\mathcal{A} = \{x_n \in \text{conv}(\mathcal{X})\}$. The operation in (22) can be done efficiently as in [27]. Algorithm 1 summarizes the above steps. The per-iteration complexity of our algorithm is $\mathcal{O}(N^2 U^2)$, and that of GDA [16] is $\mathcal{O}(N^2 U^2 + NK)$, where $N \gg U$. Note that the complexity is at least quadratic in N if the gradient is used.

The above algorithm can be extended to the weighted sum-rate maximization problem. Our algorithm can also be extended to the active IRS case [28]. In this case, we perform the proposed algorithm and update the amplification factor as

TABLE I
RUNNING TIME OF DIFFERENT METHODS WHEN $N = 500$.

| Method | Running Time (second) | | |
|--------------------|-----------------------|----------|----------|
| | $U = 5$ | $U = 10$ | $U = 15$ |
| Proposed Algorithm | 91.1 | 115.3 | 151 |
| GDA [16] | 98.8 | 171.3 | 221.2 |
| Riemannian [21] | 96.9 | 153.4 | 194.5 |

in [28] in an alternating fashion.

V. SIMULATION RESULTS

Fig. 3 shows the considered IRS-assisted multi-user downlink network at 2.6 GHz. The transmit antennas are arranged as a uniform linear array while the REs are arranged as a uniform planar array, with half-wavelength spacing. The pathloss model follows [29]. We adopt the Rician fading model [30] with the Rician factor $k = 10$.

Let $P = 30$ dBm, $\sigma_u^2 = -90$ dBm, $K \in \{2, 4\}$, and $M = 64$. The users are randomly distributed in the shaded area as shown in Fig. 3. The GDA method in [16] and the Riemannian method in [21] are the benchmarks; the solution of the Riemannian method is rounded to the discrete set. Note that our algorithm contains the outer iteration that updates \mathbf{W} iteratively as well as the inner loop that updates \mathbf{x} and \mathbf{y} iteratively under the current \mathbf{W} . We run 1000 inner iterations per outer iteration. At the beginning of each outer iteration, λ is initialized to 0.0001, and then is multiplied by 10 every after 100 inner iterations. Moreover, initialize both α and β to 0.01, and update them as $\alpha^{t+1} = 0.997\alpha^t$ and $\beta^{t+1} = 0.997\beta^t$.

Fig. 4 shows the minimum SINR versus the number of users U when $N = 500$. Observe that the minimum SINRs achieved by the different algorithms all decrease with U ; this implies that it is increasingly difficult to coordinate the beams when more users are in the network. Observed also that our algorithm always outperforms the GDA and Riemannian algorithms. Fig. 5 shows the minimum SINR versus the number of REs N when $U = 9$. As one can expect, the minimum SINRs achieved by all the algorithms increase with N . Moreover, we compare the proposed algorithm and the exhaustive search method for a toy model with $N = 10$ and $U = 5$. Our simulation shows that the proposed algorithm

attains the minimum SINR of 7.31 dB while the global optimum achieved by the exhaustive search equals 7.41 dB.

Moreover, we compare the convergence behaviors of the different algorithms in Fig. 6. Observe that our algorithm has a much faster convergence. Note that the convergence rate of GDA is sensitive to K . Finally, observe from Table I that our algorithm has the highest time-efficiency.

VI. CONCLUSION

This letter aims to maximize the minimum SINR among multiple users in an IRS-assisted downlink network under a *discrete* constraint on passive beamforming. While many existing works adopt a unit-circle relaxation, this letter proposes a convex-hull relaxation so that the new continuous problem is guaranteed to be equivalent to the original discrete problem. The relaxed problem can be efficiently solved by alternating projection/proximal gradient descent and ascent algorithm.

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